Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Departement of Computer Science
Markus Püschel, David Steurer

24. September 2018

Algorithms & Data Structures — Homework 1 — HS 18

Exercise Class (Room & TA): ______________________

Submitted by: ______________________

Peer Feedback by: ______________________

Points: _____

Exercise 1.1 *Induction.*

1. Prove via mathematical induction, that the following holds for any positive integer n:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}\,.$$

- **Base Case.**
  Let $n = 1$. Then:

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \frac{1^2 \cdot 2^2}{4} = \frac{1^2(1+1)^2}{4}$$

- **Induction Hypothesis.**
  Assume that the property holds for some postiive integer k. That is:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

- **Inductive Step.**

  We must show that the property holds for $k+1$. Add $(k+1)^2$ to both sides of our inductive hypothesis.

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \Big(\sum_{i=1}^{k} i^3\Big) + (k+1)^3 \\ &\overset{I.H.}{=} \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4} \\ &= \frac{(k^2+4k+4)(k+1)^2}{4} = \frac{(k+2)^2(k+1)^2}{4} = \frac{(k+1)^2((k+1)+1)^2}{4}.\end{aligned}$$

  By the principle of mathematical induction, this is true for any positive integer n.

2. Prove via mathematical induction that for any positive integer n,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \,.$$

- **Base Case.**
Let $n = 1$. Then $(1+x)^1 = \binom{1}{0}x^0 + \binom{1}{1}x^1 = \sum_{i=0}^{n} \binom{n}{i} x^i$.

- **Induction Hypothesis.**
Assume that the property holds for some positive integer k. That is:

$$(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i.$$

- **Inductive Step.**
We must show that the property holds for $k+1$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\overset{I.H.}{=} (1+x)\sum_{i=0}^{k} \binom{k}{i} x^i \\
&= \Big(\sum_{i=0}^{k} \binom{k}{i} x^i\Big) + \Big(\sum_{i=0}^{k} \binom{k}{i} x^{i+1}\Big) \\
&= \Big(\sum_{i=0}^{k} \binom{k}{i} x^i\Big) + \Big(\sum_{i=1}^{k+1} \binom{k}{i-1} x^i\Big) \\
&= \binom{k}{0} x^0 + \sum_{i=1}^{k} \Big(\binom{k}{i} x^i + \binom{k}{i-1} x^i\Big) + \binom{k}{k} x^{k+1} \\
&= \binom{k+1}{0} x^0 + \sum_{i=1}^{k} \binom{k+1}{i} x^i + \binom{k+1}{k+1} x^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} x^i.
\end{aligned}$$

By the principle of mathematical induction, this is true for any positive integer n.

Exercise 1.2 *Acyclic Graphs.*

Definitions:

- A graph is **acyclic** if there are no cycles. A **cycle** is a nontrivial path from vertex a to itself.
- A graph is **connected** if there is a path between every pair of vertices.
- An acyclic graph is called **non-trivial** if it has at least one edge.

For a given connected acyclic graph $G = (V, E)$, avoid using induction and prove the following:

1. There is a unique path between any pair of vertices u and v, such that $u \neq v$.

   Solution: Suppose there are two different paths between u and v. Let x be the first place they diverge. Let y be the next place they meet. Then there are two disjoint subpaths between x and y which is a cycle. This contradicts the acyclic assumption. Thus there is only one path.

2. Adding an edge between any pair of vertices creates a cycle.

   Solution: There is a unique path between u and v already proven by (1). Adding in an edge to u and v will from the cycle of that path plus this new edge.

3. Show that any non-trivial acyclic graph has at least two vertices of degree 1.
   Hint: *consider some longest path.*

   Solution: Let G be a non-trivial acyclic graph. Consider some longest path $P = v_1, v_2, \ldots, v_m$ in G. This path exists since the set of paths is not empty (because G has at least one edge) and lengths of paths are bounded by the number of vertices in G (because G is acyclic).

   Let's prove that vertices v_1 and v_m have degree 1 in G. Assume without loss of generality that $\deg v_1 > 1$ (it cannot be 0 since v_1 has a neighbour v_2). It means that v_1 has at least two different neighbours, so there exists a neighbour u of v_1 different from v_2.

   Case 1 If u belongs to P, then $u = v_i$, where $2 < i \leq m$, and $v_i, v_1, v_2, \ldots, v_i$ is a cycle in G, which contradicts the fact that G is acyclic.

   Case 2 Otherwise, $u, v_1, \ldots, v_m$ is a path in G of length $m + 1$, which contradicts the fact that P is a longest path in G. Hence $\deg v_1 = 1$.

   Using the same argument one can show that $\deg v_m = 1$. Therefore, any non-trivial acyclic graph has at least two vertices of degree 1.

Exercise 1.3 *Number of Edges in Graphs* **(1 point)**.

Definition:

- A **Tree** is an acyclic graph that is connected.

1. Show by mathematical induction that the number of edges in a tree with n vertices is $n - 1$.

   - **Base Case:** Let $n = 1$. There is a single node, and there cannot be any edge from it to itself because then there would be a cycle. There are no other nodes to connect, so there must be 0 edges.
   - **Induction Hypothesis:** Assume that any tree with k vertices has exactly $k - 1$ edges.
   - **Inductive Step:** Suppose we are given a tree with $k + 1$ vertices. Remove any vertex of degree 1 from the tree. There must be such a vertex to remove because the tree is acyclic and connected and non-trivial (See Exercise 1.2.4). This results in a tree with k vertices. By the induction hypothesis, this tree has $k - 1$ edges. Add the leaf node back to the tree. This adds only one edge to the tree, since the leaf has no children, and each node has in-degree equal to one. Thus the full tree has k edges.

     By the principle of mathematical induction, a tree with n vertices has $n - 1$ edges for any positive integer n.

2. Prove or disprove that every graph with n vertices and $n - 1$ edges is a tree.
   Solution: Consider the following graph: label the vertices with indices $\{1, \ldots, n\}$. For $i \in 2, ..., n - 1$, place an edge from vertex i to vertex $i - 1$, and place an edge from vertex $n - 1$ to vertex 1. As this is a cycle, there are n vertices and there are $n - 1$ edges, this is a counter-example, and disproves the proposition.

Exercise 1.4 *Bipartite Graphs* **(2 points)**.

1. Consider the following lemma: *If G is a bipartite graph and the bipartition of G is X and Y, then*

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v) \tag{1}$$

Then, use the lemma to prove that you can not cover the area in Figure 1, with the given tiles of size 1×2 and 2×1, depicted in the same figure.

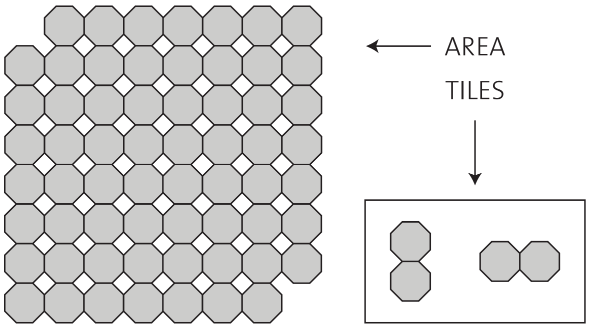

Figure 1: Cover the area with the given tiles

Solution: We can think of the area as a standard 8×8 chessboard such that the top left square and the bottom right square have been removed. We create a bipartite graph $G = (V, E)$ from the chess board. Let each white square be a vertex in X and let each black square be a vertex in Y. Note that each tile must cover exactly one black and one white square and so we will connect a vertex in X to a vertex in Y if a tile covers both of them. Because we have removed the upper left and the lower right square, which are both either white or black, we have that, without loss of generality, $|X| = 30$ and $|Y| = 32$. Now suppose that each of the squares was able to be covered. Then each vertex has degree exactly one, since it is covered by one tile. This means that $deg(v) = 1$ for all $v \in V$. Consequently:

$$\sum_{v \in X} \deg(v) = |X| \text{ and } \sum_{v \in Y} \deg(v) = |Y| \tag{2}$$

Finally, taking Lemma 1, we have:

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v) \Rightarrow |X| = |Y| \tag{3}$$

Which is a contradiction. Thus you can not cover the area with the given tiles.

2. Coloring Bipartite Graphs

Suppose you are given a map with n vertical lines. The areas of the map (i.e. the areas between the lines) must be colored such that any two neighboring areas have different colors. Prove by mathematical induction that any such map can be colored with exactly two colors. Hint: Suppose you start with a map with two vertical lines, dividing the map into three regions, colored red, blue, and red from right to left. What happens if you draw a vertical line through the blue region?

How can you modify the colors of the regions to maintain the property that neighboring regions have different colors?

- **Base Case** Let $n = 1$. Then there are two areas (one on each side of the line). Color one area red and the other blue.
- **Induction Hypothesis** Assume that for any map with k vertical lines, it is possible to color the map with only two colors.
- **Inductive Step** We now have a map with $k + 1$ vertical lines. We must show that we can color the map as described with only two colors. Remove any line from the map. There are now k vertical lines, and we can color the map according to the induction hypothesis. Next we add the line that we previously removed from the graph. The line now separates two areas that have the same color. To fix this, we can invert the colors of the areas to the left of the line.

  By the principle of mathematical induction, this is true for any positive integer n.

Exercise 1.5 *Sudoku.*

The classic Sudoku game involves a 9×9 grid. This grid is divided into nine 3×3 nonoverlapping subgrids, called blocks. The grid is partially filled by digits from 1 to 9. The objective is to fill this grid with digits so that each column, each row, and each block contains all of the digits from 1 to 9. Each digit can only appear once in a row, column or block (see Figure 2).

8			1	5		6		
			3				4	1
5						7		
					9		6	2
				3				
1	4		8					
		8						9
2	9				1			
		5		9	7			6

(a) A Sudoku puzzle

8	7	4	1	5	2	6	9	3
6	2	9	3	7	8	5	4	1
5	3	1	9	6	4	7	2	8
3	5	7	4	1	9	8	6	2
9	8	2	7	3	6	1	5	4
1	4	6	8	2	5	9	3	7
7	6	8	5	4	3	2	1	9
2	9	3	6	8	1	4	7	5
4	1	5	2	9	7	3	8	6

(b) Solution

Figure 2: Sudoku

Model this as a graph problem: give a precise definition of the graph involved and state the specific question about this graph that needs to be answered. What is the maximum vertex degree of this graph?

Solution: The graph G has 81 vertices, one vertex for each cell. Two distinct vertices u and v are connected by an edge if and only if at least one of the following conditions holds:

1. the cells u and v are in the same row.
2. the cells u and v are in the same column.
3. the cells u and v are in the same block.

The aim is to construct a 9-coloring of G, given a partial 9-coloring (defined on some subset of vertices).

Maximum vertex degree of G is 20. In fact, all vertices have degree 20. Indeed, for each vertex u there are 8 neighbours of u in the row which contains u, 8 neighbours of u in the column which contains u, and 4 neighbours of u in the block which contains u (excluding neighbours in the same row/column as u).

Submission: On Monday, 1.10.2018, hand in your solution to your TA *before* the exercise class starts.